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Stopping times and related Itô's calculus with G-Brownian motion

Xinpeng Li*, Shige Peng

School of Mathematics, Shandong University, 250100, Jinan, China

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Abstract

Under the framework of G-expectation and G-Brownian motion, we introduce Itô's integral for stochastic processes without assuming quasi-continuity. Then we can obtain Itô's integral on stopping time interval. This new formulation permits us to obtain Itô's formula for a general $C^{1,2}$-function, which essentially generalizes the previous results of Peng (2006, 2008, 2009, 2010, 2010) [21–25] as well as those of Gao (2009) [8] and Zhang et al. (2010) [27].

MSC: 60H05; 60J65

Keywords: G-Brownian motion; Stopping time; Itô's integral; Itô's formula

1. Introduction

A G-Brownian motion is a continuous process with independent and stationary increments under a nonlinear expectation called G-expectation. We first recall the notion of nonlinear expectation introduced in the study of nonlinear pricing and risk measuring problem in finance. The notion of g-expectation was introduced via BSDE in [18]. It is an ideal framework for the valuation of randomness and risk in the case of the uncertainty of probability models; see [2,7,19].

One important limitation of g-expectation is that the involved uncertain probability measures have to be absolutely continuous with respect to a reference probability measure, e.g., a Wiener

* Corresponding author.
E-mail addresses: xplisdu@gmail.com (X. Li), peng@sdu.edu.cn (S. Peng).

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measure. But for the well-known problem of volatility model uncertainty (uncertain volatility model, abbreviated UVM) in finance, there is an uncountable number of unknown probabilities which are essentially singular from each other. Avellaneda et al. [1] and Lyons [16] studied this UVM problem for the situation of state-dependent options. The situation of path-dependence is more challenging and needed to create a new framework more general than the classical notion of probability.

Such types of fully nonlinear expectations for situations of path-dependence were constructed by Peng (2004, 2005) in [19,20] where two very different approaches were introduced to solve the involved problem of dynamic consistency. The first one is a generalized dynamic programming principle for the path-dependent situation. The second one is about to use the notion of nonlinear monotone semigroups of Nisio's type, called nonlinear Markov chain, to develop a nonlinear version of Kolmogorov consistency theorem in order to construct nonlinear expectation spaces which play the role of the classical probability spaces.

The most typical example of the above-mentioned fully nonlinear expectation is the G-expectation under which the corresponding canonical path is called G-Brownian motion. This notion and the corresponding stochastic calculus of Itô's type were introduced and systematically developed in [21–25].

Independently, Denis and Martini (2006) [6] defined a stochastic integral of Itô's type using their deep results of quasi-surely analysis to solve the hedging problem of UVM. It was proved in [5] that the basic space $L_G^2(\Omega)$ defined in [21–25] (see the next section) coincides with that in [6].

More precisely, G-Brownian motion is a continuous process $(B_t)_{t\geq 0}$ defined on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ with independent and stationary increments. It was proved that each increment $X = B_{t+s} - B_t$ of B is G-normally distributed, namely,

$$aX + b\bar{X} \overset{d}{=} \sqrt{a^2+b^2}X, \quad \forall a, b \geq 0,$$

where $\bar{X}$ is an independent copy of X under $\hat{\mathbb{E}}$. A new type of stochastic integral and the related Itô's calculus has been introduced in [21] (see also [22–25]).

For example, if Φ is a C^2-function such that Φ_{xx} satisfies polynomial growth condition, then we have

$$\Phi(B_t) - \Phi(B_s) = \int_s^t \Phi_x(B_u)\mathrm{d}B_u + \frac{1}{2}\int_s^t \Phi_{xx}(B_u)\mathrm{d}\langle B\rangle_u. \tag{1}$$

An interesting problem is how to extend the above formulation to the situation where Φ is simply a C^2-function. The main obstacle to treat this situation is that the notion of stopping times and the related properties have not yet been well understood and studied within the framework of G-expectation and G-Brownian motion. A difficulty hidden behind is that until now the theory is mainly based on the space of random variables $X = X(\omega)$ which are quasi-continuous with respect to the natural Choquet capacity $\hat{c}(\cdot)$ of G-expectation. It is not yet clear that the martingale properties still hold for random variables without quasi-continuous condition. On the other hand, stopping times are closely related to random variables without quasi-continuity properties. Recently, Gao [8] and Zhang et al. [27] have improved Itô's formula of Peng [21–23]. But the problem of (1) for C^2-function without the growth restriction is still open.

In this paper, we will overcome this restriction on Φ by introducing Itô's stochastic integrals $\int_0^t \eta_s \mathrm{d}B_s$ where, for each t, the integrand η_t needs not to be a quasi-continuous random variable. Within this framework, we can treat the fundamentally important Itô's integral $\int_0^{t\wedge\tau} \eta_s \mathrm{d}B_s$ for a stopping time τ and then obtain some important properties for the related stochastic calculus. A very general form of Itô's formula with respect to G-Brownian motion, which is comparable with that from the classical Itô's calculus, has been obtained. In particular, (1) is proved to be true for $\Phi \in C^2$.

This paper is organized as follows. In the next section, we recall some basic notions and results of G-Brownian motion under a G-expectation and the related space of random variables. In Section 3, we introduce a new space $M_*^2(0, T)$ of stochastic processes which are not necessarily quasi-continuous and then define the related Itô's integral on this space. In Section 4, we discuss Itô's integral defined on $[0, \tau]$ where τ is a stopping time. This allows us to have a Itô's integral for a space $M_\omega^2(0, T)$ which is larger than $M_*^2(0, T)$. Finally in Section 5, we prove the above-mentioned general form of Itô's formula.

We believe that some notions and properties of this paper will become important and basic tools in the further development of G-Brownian motion and the corresponding nonlinear expectation analysis.

2. Basic settings

We first present some preliminaries in the theory of sublinear expectations and the related G-Brownian motions. More details can be found in [21–25]. Some further developments and related topics can be found in [3,5,6,8–14,17–20,26].

Definition 2.1. Let Ω be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on Ω with $c \in \mathcal{H}$ for all constants c, and $|X| \in \mathcal{H}$, if $X \in \mathcal{H}$. $\mathcal{H}$ is considered as the space of our "random variables". A sublinear expectation $\mathbb{E}$ on $\mathcal{H}$ is a functional $\mathbb{E} : \mathcal{H} \mapsto \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(1) Monotonicity: $\mathbb{E}[X] \geq \mathbb{E}[Y]$, if $X \geq Y$.
(2) Constant preserving: $\mathbb{E}[c] = c,\ \forall c \in \mathbb{R}$.
(3) Sub-additivity: $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$.
(4) Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X],\ \forall \lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sublinear expectation space. $X \in \mathcal{H}$ is called a random variable on $(\Omega, \mathcal{H})$. We often call $Y = (Y_1, \ldots, Y_d)$, $Y_i \in \mathcal{H}$ a d-dimensional random vector on $(\Omega, \mathcal{H})$.

Let us consider a space of random variables $\mathcal{H}$ satisfying: if $X_i \in \mathcal{H},\ i = 1, \ldots, d$, then

$$\varphi(X_1, \ldots, X_d) \in \mathcal{H}, \quad \forall \varphi \in C_{b,\,\mathrm{Lip}}(\mathbb{R}^d),$$

where $C_{b,\,\mathrm{Lip}}(\mathbb{R}^d)$ is the space of all bounded and Lipschitz continuous functions on $\mathbb{R}^d$.

Definition 2.2. An m-dimensional random vector $Y = (Y_1, \ldots, Y_m)$ is said to be independent of another n-dimensional random vector $X = (X_1, \ldots, X_n)$ if for each $\varphi \in C_{b,\,\mathrm{Lip}}(\mathbb{R}^n \times \mathbb{R}^m)$,

$$\mathbb{E}[\varphi(X, Y)] = \mathbb{E}[\mathbb{E}[\varphi(x, Y)]_{x=X}].$$

Let X_1 and X_2 be two n-dimensional random vectors defined respectively on sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if for each $\varphi \in C_{b,\,\mathrm{Lip}}(\mathbb{R}^n)$,

$$\mathbb{E}_1[\varphi(X_1)] = \mathbb{E}_2[\varphi(X_2)].$$

If $X, \bar{X}$ are two n-dimensional random vectors on $(\Omega, \mathcal{H}, \mathbb{E})$ and $\bar{X}$ is identically distributed with X and independent of X, then $\bar{X}$ is said to be an independent copy of X.

Definition 2.3 (*G-Normal Distribution*). A d-dimensional random vector $X = (X_1, \ldots, X_d)$ on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called G-normally distributed if for each $a, b \geq 0$, we have

$$aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2}X, \tag{2}$$

where $\bar{X}$ is an independent copy of X. Here the letter G denotes the function

$$G(A) := \frac{1}{2}\mathbb{E}[(AX, X)] : \mathbb{S}_d \mapsto \mathbb{R}.$$

It is also proved in [22] that, for each $\mathbf{a} \in \mathbb{R}^d$ and $p \in [1, \infty)$

$$\mathbb{E}[|(\mathbf{a}, X)|^p] = \frac{1}{\sqrt{2\pi\sigma^2_{\mathbf{a}\mathbf{a}^T}}} \int_{-\infty}^{\infty} |x|^p \exp\left(\frac{-x^2}{2\sigma^2_{\mathbf{a}\mathbf{a}^T}}\right) \mathrm{d}x,$$

where $\sigma^2_{\mathbf{a}\mathbf{a}^T} = 2G(\mathbf{a}\mathbf{a}^T)$.

Definition 2.4. A d-dimensional stochastic process $\xi_t(\omega) = (\xi_t^1, \ldots, \xi_t^d)(\omega)$ defined on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is a family of d-dimensional random vectors ξ_t parameterized by $t \in [0, \infty)$ such that $\xi_t^i \in \mathcal{H}$, for each $i = 1, \ldots, d$ and $t \in [0, \infty)$.

The most typical stochastic process on a sublinear expectation space is the so-called G-Brownian motion (see [21–25]).

Definition 2.5 (*G-Brownian Motion*). Let $G : \mathbb{S}_d \mapsto \mathbb{R}$ be a given monotonic and sublinear function. A process $\{B_t(\omega)\}_{t\geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called a G-Brownian motion if for each $n \in \mathbb{N}$ and $0 \leq t_1 < \cdots < t_n < \infty$, $B_{t_1}, \ldots, B_{t_n} \in \mathcal{H}$ and the following properties are satisfied:

(1) $B_0(\omega) = 0$.
(2) For each $t, s \geq 0$, the increment $B_{t+s} - B_t$ is independent of $(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$, for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$.
(3) $\mathbb{E}[B_t] = \mathbb{E}[-B_t] = 0$, $B_{t+s} - B_t \overset{d}{=} B_s$, $t, s \geq 0$.
(4) $\mathbb{E}[|B_t|^3]/t \to 0$ as $t \to 0$.

Remark 2.6. Conditions (3) and (4) can be equivalently replaced by the following one (see [24]): $(3')$ $B_{t+s} - B_t$ is G-normally distributed. Condition (4) is to guarantee that B has continuous paths. Without this condition B becomes a G-Lévy process; see [11].

It is not difficult to construct a G-Brownian motion. Let $\Omega = C_0^d(\mathbb{R}^+)$, i.e., the space of all $\mathbb{R}^d$-valued continuous paths $(\omega_t)_{t\in\mathbb{R}^+}$ with $\omega_0 = 0$. The corresponding canonical process is $B_t(\omega) = \omega_t,\ t \in [0, \infty)$. We let $\mathcal{H} := \mathrm{Lip}(\Omega)$ to be our linear space of random variables, where $\mathrm{Lip}(\Omega)$ denotes the spaces of Lipschitzian cylinder functions on Ω, namely,

$$\mathrm{Lip}(\Omega) := \{\varphi(\omega_{t_1}, \ldots, \omega_{t_n}) : \ \forall n \geq 1, t_1, \ldots, t_n \in [0, \infty),\ \forall \varphi \in C_{b,\,\mathrm{Lip}}(\mathbb{R}^{d\times n})\}.$$

We now construct a sublinear expectation $\hat{\mathbb{E}}$ on $(\Omega, \mathcal{H})$ as follows. For each $\xi \in \mathcal{H} = \mathrm{Lip}(\Omega)$ of the form

$$\xi = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}), \quad t_1 < t_2 < \cdots < t_n,$$

we set

$$\hat{\mathbb{E}}[\xi] := \bar{\mathbb{E}}[\varphi(X_1\sqrt{t_1}, X_2\sqrt{t_2 - t_1}, \ldots, X_n\sqrt{t_n - t_{n-1}})],$$

where $\{X_i\}_{i=1}^{\infty}$ is a given sequence of random variables in a sublinear expectation space $(\bar{\Omega}, \bar{\mathcal{H}}, \bar{\mathbb{E}})$ such that, for each i, $X_i \overset{d}{=} X_1$, X_{i+1} is independent of $(X_1, \ldots, X_i)$ and X_1 is G-normally distributed. It is easy to check that we have consistently defined a sublinear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$. Moreover, under this $\hat{\mathbb{E}}$, the canonical process $(B_t)_{t\geq 0}$ becomes a G-Brownian motion. We denote by $L_G^p(\Omega)$ the completion of $\mathrm{Lip}(\Omega)$ under the natural norm $\|X\|_p := \hat{\mathbb{E}}[|X|^p]^{1/p}$.

We denote by $\mathcal{B}(\Omega)$ the Borel σ-algebra of Ω. It was proved in [5] (see also [12] for a simple proof) that there exists a weakly compact family $\mathcal{P}$ of probability measures defined on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\hat{\mathbb{E}}[X] = \sup_{P\in\mathcal{P}} E_P[X], \quad \forall X \in \mathrm{Lip}(\Omega).$$

We will use a deep result of quasi-surely analysis in [6]. We introduce the natural Choquet capacity (see [4]) associated to $\mathcal{P}$:

$$\hat{c}(A) := \sup_{P\in\mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$

We denote, for each $t \in [0, \infty)$:

- $\Omega_t := \{\omega_{\cdot\wedge t} : \omega \in \Omega\}$,
- $\mathcal{F}_t := \mathcal{B}(\Omega_t)$,
- $L^0(\Omega)$: the space of all $\mathcal{B}(\Omega)$-measurable real functions,
- $L^0(\Omega_t)$: the space of all $\mathcal{B}(\Omega_t)$-measurable real functions,
- $B_b(\Omega)$: all bounded elements in $L^0(\Omega)$; $B_b(\Omega_t) := B_b(\Omega) \cap L^0(\Omega_t)$,
- $C_b(\Omega)$: all continuous elements in $B_b(\Omega)$; $C_b(\Omega_t) := C_b(\Omega) \cap L^0(\Omega_t)$.

We can extend the domain of G-expectation $\hat{\mathbb{E}}$ from $L_G^1(\Omega)$ to the space of random variables $X \in L^0(\Omega)$ such that $E_P[X]$ exists for each $P \in \mathcal{P}$ by setting $\hat{\mathbb{E}}[X] := \sup_{P\in\mathcal{P}} E_P[X]$.

Definition 2.7. A set $A \subset \Omega$ is polar if $\hat{c}(A) = 0$. A property holds "quasi-surely" (q.s. for short) if it holds outside a polar set.

A real function X on Ω is said to be quasi-continuous (q.c. for short) if for each $\varepsilon > 0$, there exists an open set O with $\hat{c}(O) < \varepsilon$ such that $X|_{O^c}$ is continuous. Let X and Y be two random variables, we say that X is a version of Y, if $X = Y$ q.s. Then the space $L_G^p(\Omega)$ is the completion of $C_b(\Omega)$ under the natural norm $\|\cdot\|_p$ and (Theorem 52 in [5])

$$L_G^p = \{X \in L^0(\Omega) : X \text{ has a q.c. version and } \lim_{n\to\infty} \hat{\mathbb{E}}[|X|^p \mathbf{I}_{\{|X|>n\}}] = 0\}.$$

We denote by $L_*^p(\Omega)$ (respectively $L_*^p(\Omega_t)$) the completion of $B_b(\Omega)$ (respectively $B_b(\Omega_t)$) under the norm $\|\cdot\|_p$. Then we have $L_*^p(\Omega) \supset L_G^p(\Omega)$ and the following propositions (Propositions 17 and 18 in [5]).

Proposition 2.8. *For a given $p \in (0, +\infty]$, let $\{X_n\}_{n=1}^{\infty}$ be a sequence in $L_*^p(\Omega)$ which converges to X in $L_*^p(\Omega)$. Then there exists a subsequence (X_{n_k}) which converges to X quasi-surely in the sense that it converges to X outside a polar set.*

Proposition 2.9. *For each $p > 0$,*

$$L_*^p(\Omega) = \{X \in L^0(\Omega) : \lim_{n\to\infty} \hat{\mathbb{E}}[|X|^p \mathbf{I}_{\{|X|>n\}}] = 0\}.$$

For notational simplicity, we denote by $B^i := B^{\mathbf{e}_i}$ the ith coordinate of the d-dimensional G-Brownian motion B, under a given orthonormal basis $(\mathbf{e}_1, \ldots, \mathbf{e}_d)$ of $\mathbb{R}^d$. We also denote by $B_t^{\mathbf{a}} := (\mathbf{a}, B_t)$ for fixed $\mathbf{a} \in \mathbb{R}^d$. Then $(B_t^{\mathbf{a}})_{t\geq 0}$ is a 1-dimensional G-Brownian motion with $\sigma^2_{\mathbf{aa}^T} = \hat{\mathbb{E}}[(\mathbf{a}, B_1)^2]$ and $\sigma^2_{-\mathbf{aa}^T} = -\hat{\mathbb{E}}[-(\mathbf{a}, B_1)^2]$.

We introduce the following properties, which are important in our stochastic calculus.

Proposition 2.10. *For each $0 \leq t < T$, $\xi \in L_*^2(\Omega_t)$, we have*

$$\hat{\mathbb{E}}[\xi(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})] = 0.$$

Proof. Let $P \in \mathcal{P}$ be given. If $\xi \in C_b(\Omega_t)$, then we have

$$0 = -\hat{\mathbb{E}}[-\xi(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})] \leq E_P[\xi(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})] \leq \hat{\mathbb{E}}[\xi(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})] = 0.$$

In the case when $\xi \in L_*^2(\Omega_t)$, we have $E_P[|\xi|^2] \leq \hat{\mathbb{E}}[|\xi|^2] < \infty$. Since it is known that $C_b(\Omega_t)$ is dense in $L^2(\Omega, \mathcal{F}_t, P)$, we can choose a sequence $\{\xi_n\}_{n=1}^{\infty}$ in $C_b(\Omega_t)$ such that $E_P[|\xi - \xi_n|^2] \to 0$. Thus

$$E_P[\xi(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})] = \lim_{n\to\infty} E_P[\xi_n(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})] = 0. \quad \square$$

Proposition 2.11. *For each $0 \leq t < T$, $\xi \in B_b(\Omega_t)$, we have*

$$\hat{\mathbb{E}}[\xi^2(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})^2 - \sigma^2_{\mathbf{aa}^T}\xi^2(T-t)] \leq 0. \tag{3}$$

Proof. If $\xi \in C_b(\Omega_t)$, then by Lemma 3.5 in [24], we have

$$\hat{\mathbb{E}}[\xi^2(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})^2 - \sigma^2_{\mathbf{aa}^T}\xi^2(T-t)] = 0.$$

Thus (3) holds for $\xi \in C_b(\Omega_t)$. It follows that, for each fixed $P \in \mathcal{P}$, we have

$$E_P[\xi^2(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})^2 - \sigma^2_{\mathbf{aa}^T}\xi^2(T-t)] \leq 0. \tag{4}$$

In the case when $\xi \in B_b(\Omega_t)$, we can find a sequence $\{\xi_n\}_{n=1}^{\infty}$ in $C_b(\Omega_t)$, such that $\xi_n \to \xi$ in $L^p(\Omega, \mathcal{F}_t, P)$, for some $p > 2$. Thus we have

$$E_P[\xi_n^2(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})^2 - \sigma^2_{\mathbf{aa}^T}\xi_n^2(T-t)] \leq 0,$$

and then, by letting $n \to \infty$, obtain (4) for $\xi \in B_b(\Omega_t)$. Thus (3) follows immediately for $\xi \in B_b(\Omega_t)$. $\square$

3. A generalized Itô's integral

In [21–25] as well as [8,27], their work is mainly based on the space $L_G^p(\Omega)$, i.e., the random variables are quasi-continuous. Thanks to Propositions 2.10 and 2.11, we use $L_*^p(\Omega)$, instead of $L_G^p(\Omega)$ to generalize our Itô's integral on a larger space of stochastic processes $M_*^2(0,T)$ defined as follows. For $p \geq 1$ and $T \in \mathbb{R}^+$ fixed, we first consider the following type of simple processes:

$$M_{b,0}(0,T) = \left\{ \eta : \eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) \mathbf{I}_{[t_j,t_{j+1})}(t), \right.$$
$$\left. \forall N \in \mathbb{N}, 0 = t_0 < \cdots < t_N = T, \xi_j \in B_b(\Omega_{t_j}), j = 0, \ldots, N-1 \right\}.$$

Definition 3.1. For an $\eta \in M_{b,0}(0,T)$ with $\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) \mathbf{I}_{[t_j,\, t_{j+1})}(t)$, the related Bochner integral is

$$\int_0^T \eta_t(\omega) \mathrm{d}t := \sum_{j=0}^{N-1} \xi_j(\omega)(t_{j+1} - t_j).$$

For each $\eta \in M_{b,0}(0,T)$, we set

$$\hat{\mathbb{E}}_T[\eta] := \frac{1}{T} \hat{\mathbb{E}}\left[\int_0^T \eta_t \mathrm{d}t\right] = \frac{1}{T} \hat{\mathbb{E}}\left[\sum_{j=0}^{N-1} \xi_j(\omega)(t_{j+1} - t_j)\right].$$

Then $\hat{\mathbb{E}}_T : M_{b,0}(0,T) \mapsto \mathbb{R}$ forms a sublinear expectation.

Definition 3.2. For each $p \geq 1$, we denote by $M_*^p(0,T)$ the completion of $M_{b,0}(0,T)$ under the norm

$$\|\eta\|_{M^p(0,T)} = \left\{ \hat{\mathbb{E}}\left[\int_0^T |\eta_t|^p \mathrm{d}t\right] \right\}^{1/p}.$$

We have $M_*^p(0,T) \supset M_*^q(0,T)$ for $p \leq q$. The following process

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) \mathbf{I}_{[t_j,t_{j+1})}(t), \quad \xi_j \in L_*^p(\Omega_{t_j}),\ j = 0, \ldots, N-1$$

is also in $M_*^p(0,T)$.

Definition 3.3. For each $\eta \in M_{b,0}(0,T)$ with the form

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) \mathbf{I}_{[t_j,t_{j+1})}(t),$$

we define Itô's integral

$$I(\eta) = \int_0^T \eta_t \mathrm{d}B_t^{\mathbf{a}} := \sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}}^{\mathbf{a}} - B_{t_j}^{\mathbf{a}}).$$

Lemma 3.4. *The mapping $I : M_{b,0}(0,T) \to L_*^2(\Omega_T)$ is a linear continuous mapping and thus can be continuously extended to $I : M_*^2(0,T) \to L_*^2(\Omega_T)$. We have*

$$\hat{\mathbb{E}}\left[\int_0^T \eta_t \mathrm{d}B_t^{\mathbf{a}}\right] = 0, \tag{5}$$

$$\hat{\mathbb{E}}\left[\left(\int_0^T \eta_t \mathrm{d}B_t^{\mathbf{a}}\right)^2\right] \leq \sigma^2_{\mathbf{aa}^T}\hat{\mathbb{E}}\left[\int_0^T \eta_t^2 \mathrm{d}t\right]. \tag{6}$$

Proof. We only need to prove (5) and (6) hold for $\eta \in M_{b,0}(0,T)$. From Proposition 2.10, for each j,

$$\hat{\mathbb{E}}[\xi_j(B^{\mathbf{a}}_{t_{j+1}} - B^{\mathbf{a}}_{t_j})] = \hat{\mathbb{E}}[-\xi_j(B^{\mathbf{a}}_{t_{j+1}} - B^{\mathbf{a}}_{t_j})] = 0.$$

Thus we have

$$\begin{aligned}\hat{\mathbb{E}}\left[\int_0^T \eta_s \mathrm{d}B_s^{\mathbf{a}}\right] &= \hat{\mathbb{E}}\left[\int_0^{t_{N-1}} \eta_s \mathrm{d}B_s^{\mathbf{a}} + \xi_{N-1}(B^{\mathbf{a}}_{t_N} - B^{\mathbf{a}}_{t_{N-1}})\right]\\ &= \hat{\mathbb{E}}\left[\int_0^{t_{N-1}} \eta_s \mathrm{d}B_s^{\mathbf{a}}\right] = \cdots = \hat{\mathbb{E}}[\xi_0(B^{\mathbf{a}}_{t_1} - B^{\mathbf{a}}_{t_0})] = 0.\end{aligned}$$

We now prove (6). We first apply Proposition 2.10 to derive

$$\begin{aligned}\hat{\mathbb{E}}\left[\left(\int_0^T \eta_t \mathrm{d}B_t^{\mathbf{a}}\right)^2\right] &= \hat{\mathbb{E}}\left[\left(\int_0^{t_{N-1}} \eta_t \mathrm{d}B_t^{\mathbf{a}} + \xi_{N-1}(B^{\mathbf{a}}_{t_N} - B^{\mathbf{a}}_{t_{N-1}})\right)^2\right]\\ &= \hat{\mathbb{E}}\left[\left(\int_0^{t_{N-1}} \eta_t \mathrm{d}B_t^{\mathbf{a}}\right)^2 + \xi^2_{N-1}(B^{\mathbf{a}}_{t_N} - B^{\mathbf{a}}_{t_{N-1}})^2\right.\\ &\quad \left. +\ 2\left(\int_0^{t_{N-1}} \eta_t \mathrm{d}B_t^{\mathbf{a}}\right)\xi_{N-1}(B^{\mathbf{a}}_{t_N} - B^{\mathbf{a}}_{t_{N-1}})\right]\\ &= \hat{\mathbb{E}}\left[\left(\int_0^{t_{N-1}} \eta_t \mathrm{d}B_t^{\mathbf{a}}\right)^2 + \xi^2_{N-1}(B^{\mathbf{a}}_{t_N} - B^{\mathbf{a}}_{t_{N-1}})^2\right]\\ &= \cdots = \hat{\mathbb{E}}\left[\sum_{i=0}^{N-1}\xi_i^2(B^{\mathbf{a}}_{t_{i+1}} - B^{\mathbf{a}}_{t_i})^2\right].\end{aligned}$$

Then by Proposition 2.11, we have

$$\hat{\mathbb{E}}[\xi_i^2(B^{\mathbf{a}}_{t_{i+1}} - B^{\mathbf{a}}_{t_i})^2 - \sigma^2_{\mathbf{aa}^T}\xi_i^2(t_{i+1} - t_i)] \leq 0.$$

Thus

$$\begin{aligned}&\hat{\mathbb{E}}\left[\left(\int_0^T \eta_t \mathrm{d}B_t^{\mathbf{a}}\right)^2\right] = \hat{\mathbb{E}}\left[\sum_{i=0}^{N-1}\xi_i^2(B^{\mathbf{a}}_{t_{i+1}} - B^{\mathbf{a}}_{t_i})^2\right]\\ &\leq \hat{\mathbb{E}}\left[\sum_{i=0}^{N-1}(\xi_i^2(B^{\mathbf{a}}_{t_{i+1}} - B^{\mathbf{a}}_{t_i})^2 - \sigma^2_{\mathbf{aa}^T}\xi_i^2(t_{i+1} - t_i))\right] + \hat{\mathbb{E}}\left[\sum_{i=0}^{N-1}\sigma^2_{\mathbf{aa}^T}\xi_i^2(t_{i+1} - t_i)\right]\\ &\leq \sum_{i=0}^{N-1}\hat{\mathbb{E}}[\xi_i^2(B^{\mathbf{a}}_{t_{i+1}} - B^{\mathbf{a}}_{t_i})^2 - \sigma^2_{\mathbf{aa}^T}\xi_i^2(t_{i+1} - t_i)] + \hat{\mathbb{E}}\left[\sum_{i=0}^{N-1}\sigma^2_{\mathbf{aa}^T}\xi_i^2(t_{i+1} - t_i)\right]\\ &\leq \hat{\mathbb{E}}\left[\sum_{i=0}^{N-1}\sigma^2_{\mathbf{aa}^T}\xi_i^2(t_{i+1} - t_i)\right] = \sigma^2_{\mathbf{aa}^T}\hat{\mathbb{E}}\left[\int_0^T \eta_t^2 \mathrm{d}t\right]. \quad \square\end{aligned}$$

The following proposition can be verified directly by the definition of Itô's integral.

Proposition 3.5. *Let $\eta, \theta \in M_*^2(0,T)$, and let $0 \le s \le r \le t \le T$. Then*

(1) $\int_s^t \eta_u \mathrm{d}B_u^{\mathbf{a}} = \int_s^r \eta_u \mathrm{d}B_u^{\mathbf{a}} + \int_r^t \eta_u \mathrm{d}B_u^{\mathbf{a}}$;

(2) $\int_s^t (\alpha\eta_u + \theta_u)\mathrm{d}B_u^{\mathbf{a}} = \alpha \int_s^t \eta_u \mathrm{d}B_u^{\mathbf{a}} + \int_s^t \theta_u \mathrm{d}B_u^{\mathbf{a}}$, *where $\alpha \in B_b(\Omega_s)$.*

Proposition 3.6. *For each $\eta \in M_*^2(0,T)$, we have*

$$\hat{\mathbb{E}}\left[\sup_{0\le t\le T}\left|\int_0^t \eta_s \mathrm{d}B_s^{\mathbf{a}}\right|^2\right] \le 4\sigma^2_{\mathbf{a}\mathbf{a}^T}\hat{\mathbb{E}}\left[\int_0^T \eta_s^2 \mathrm{d}s\right]. \tag{7}$$

Proof. Since for each $\alpha \in B_b(\Omega_t)$, we have

$$E_P\left[\alpha \int_t^T \eta_s \mathrm{d}B_s^{\mathbf{a}}\right] = 0,$$

thus, for each fixed $P \in \mathcal{P}$, the process $\int_0^\cdot \eta_s \mathrm{d}B_s^{\mathbf{a}}$ is a P-martingale. It follows from the classical Doob's martingale inequality that

$$\begin{aligned} E_P\left[\sup_{0\le t\le T}\left|\int_0^t \eta_s \mathrm{d}B_s^{\mathbf{a}}\right|^2\right] &\le 4E_P\left[\left|\int_0^T \eta_s \mathrm{d}B_s^{\mathbf{a}}\right|^2\right] \le 4\hat{\mathbb{E}}\left[\left|\int_0^T \eta_s \mathrm{d}B_s^{\mathbf{a}}\right|^2\right] \\ &\le 4\sigma^2_{\mathbf{a}\mathbf{a}^T}\hat{\mathbb{E}}\left[\int_0^T \eta_s^2 \mathrm{d}s\right]. \end{aligned}$$

Thus (7) holds. □

Proposition 3.7. *For any $\eta \in M_*^2(0,T)$ and $0 \le t \le T$, $\int_0^t \eta_s \mathrm{d}B_s^{\mathbf{a}}$ is continuous in t quasi-surely.*

Proof. The claim is true for $\eta \in M_{b,0}(0,T)$ since $(B_t)_{t\ge0}$ is continuous quasi-surely. In the case when $\eta \in M_*^2(0,T)$, there exists $\eta^n \in M_{b,0}(0,T)$, such that $\hat{\mathbb{E}}\left[\int_0^T (\eta_s - \eta_s^n)^2 \mathrm{d}s\right] \to 0$. By Proposition 3.6, we have

$$\hat{\mathbb{E}}\left[\sup_{0\le t\le T}\left|\int_0^t (\eta_s - \eta_s^n)\mathrm{d}B_s^{\mathbf{a}}\right|^2\right] \le 4\sigma^2_{\mathbf{a}\mathbf{a}^T}\hat{\mathbb{E}}\left[\int_0^T (\eta_s - \eta_s^n)^2 \mathrm{d}s\right] \to 0.$$

By combining this with Proposition 2.8, we see that there is some subsequence (η^{n_k}) such that, quasi-surely, the sequence of processes $\int_0^\cdot \eta_s^{n_k} \mathrm{d}B_s^{\mathbf{a}}$ uniformly converges to $\int_0^\cdot \eta_s \mathrm{d}B_s^{\mathbf{a}}$ on $[0,T]$. Thus $\int_0^t \eta_s \mathrm{d}B_s^{\mathbf{a}}$ is continuous in t quasi-surely. □

The following propositions will be applied to the proof of our new Itô's formula. They shall also find important applications in the further development of the G-stochastic analysis.

Proposition 3.8. *For each $p \ge 1$ and $X \in M_*^p(0,T)$, we have*

$$\lim_{n\to\infty}\hat{\mathbb{E}}\left[\int_0^T |X_t|^p \mathbf{I}_{\{|X_t|>n\}} \mathrm{d}t\right] = 0. \tag{8}$$

Proof. For each $X \in M_*^p(0,T)$, we can find a sequence $\{X^{(n)}\}_{n=1}^{\infty}$ in $M_{b,0}(0,T)$ such that $\hat{\mathbb{E}}\left[\int_0^T |X_t - X_t^{(n)}|^p \mathrm{d}t\right] \to 0$. Let $x_n = \sup_{\omega\in\Omega,\ t\in[0,T]} |X_t^{(n)}(\omega)|$ and $\bar{X}^{(n)} = (X \wedge x_n) \vee (-x_n)$. Since $|X - \bar{X}^{(n)}| \le |X - X^{(n)}|$, we have $\hat{\mathbb{E}}\left[\int_0^T |X_t - \bar{X}_t^{(n)}|^p \mathrm{d}t\right] \to 0$. This also implies that, for any sequence $\{\alpha_n\}$ tending to $+\infty$,

$$\lim_{n\to\infty} \hat{\mathbb{E}}\left[\int_0^T |X_t - (X_t \wedge \alpha_n) \vee (-\alpha_n)|^p \mathrm{d}t\right] = 0.$$

Now we have for all $n \in \mathbb{N}$,

$$\hat{\mathbb{E}}\left[\int_0^T |X_t|^p \mathbf{I}_{\{|X_t|>n\}} \mathrm{d}t\right] = \hat{\mathbb{E}}\left[\int_0^T (|X_t| - n + n)^p \mathbf{I}_{\{|X_t|>n\}} \mathrm{d}t\right]$$
$$\le 2^{p-1}\left(\hat{\mathbb{E}}\left[\int_0^T (|X_t| - n)^p \mathbf{I}_{\{|X_t|>n\}} \mathrm{d}t\right] + n^p \hat{\mathbb{E}}\left[\int_0^T \mathbf{I}_{\{|X_t|>n\}} \mathrm{d}t\right]\right).$$

The first term of the right-hand side tends to 0 since

$$\hat{\mathbb{E}}\left[\int_0^T (|X_t| - n)^p \mathbf{I}_{\{|X_t|>n\}} \mathrm{d}t\right] = \hat{\mathbb{E}}\left[\int_0^T |X_t - (X_t \wedge n) \vee (-n)|^p \mathrm{d}t\right] \to 0.$$

For the second term, since

$$\frac{n^p}{2^p} \mathbf{I}_{\{|X_t|>n\}} \le \left(|X_t| - \frac{n}{2}\right)^p \mathbf{I}_{\{|X_t|>n\}} \le \left(|X_t| - \frac{n}{2}\right)^p \mathbf{I}_{\{|X_t|>\frac{n}{2}\}},$$

we have

$$\frac{n^p}{2^p} \hat{\mathbb{E}}\left[\int_0^T \mathbf{I}_{\{|X_t|>n\}} \mathrm{d}t\right] \le \hat{\mathbb{E}}\left[\int_0^T \left(|X_t| - \frac{n}{2}\right)^p \mathbf{I}_{\{|X_t|>\frac{n}{2}\}} \mathrm{d}t\right] \to 0.$$

Consequently, (8) holds for $X \in M_*^p(0,T)$. $\square$

Corollary 3.9. *For each $\eta \in M_*^2(0,T)$, let $\eta_s^n = (-n) \vee (\eta_s \wedge n)$, then we have $\int_0^t \eta_s^n \mathrm{d}B_s^{\mathbf{a}} \to \int_0^t \eta_s \mathrm{d}B_s^{\mathbf{a}}$ in $L_*^2(\Omega_T)$ for each $t \le T$.*

Proposition 3.10. *Let $X \in M_*^p(0,T)$. Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $\eta \in M_*^p(0,T)$ satisfying $\hat{\mathbb{E}}\left[\int_0^T |\eta_t| \mathrm{d}t\right] \le \delta$ and $|\eta_t(\omega)| \le 1$, we have $\hat{\mathbb{E}}\left[\int_0^T |X_t|^p |\eta_t| \mathrm{d}t\right] \le \varepsilon$.*

Proof. For each $\varepsilon > 0$, by Proposition 3.8, there exists $N > 0$ such that $\hat{\mathbb{E}}\left[\int_0^T |X_t|^p \mathbf{I}_{\{|X_t|>N\}} \mathrm{d}t\right] \le \frac{\varepsilon}{2}$. Taking $\delta = \frac{\varepsilon}{2N^p}$, then we have

$$\hat{\mathbb{E}}\left[\int_0^T |X_t|^p |\eta_t| \mathrm{d}t\right] \le \hat{\mathbb{E}}\left[\int_0^T |X_t|^p |\eta_t| \mathbf{I}_{\{|X_t|>N\}} \mathrm{d}t\right] + \hat{\mathbb{E}}\left[\int_0^T |X_t|^p |\eta_t| \mathbf{I}_{\{|X_t|\le N\}} \mathrm{d}t\right]$$
$$\le \hat{\mathbb{E}}\left[\int_0^T |X_t|^p \mathbf{I}_{\{|X_t|>N\}} \mathrm{d}t\right] + N^p \hat{\mathbb{E}}\left[\int_0^T |\eta_t| \mathrm{d}t\right] \le \varepsilon. \quad \square$$

Proposition 3.11. *Let $p \ge 1$, $X, \eta \in M_*^p(0,T)$ and η is bounded, then $X\eta \in M_*^p(0,T)$.*

Proof. We can find, for $n = 1, 2, \ldots, X^n, \eta^n \in M_{b,0}(0, T)$ such that $\{\eta^n\}_{n=1}^{\infty}$ is uniformly bounded and

$$\|X - X^n\|_{M^p(0,T)} \to 0, \qquad \|\eta - \eta^n\|_{M^p(0,T)} \to 0.$$

We have

$$\hat{\mathbb{E}}\left[\int_0^T |X_t\eta_t - X_t^n\eta_t^n|^p \mathrm{d}t\right] \leq 2^{p-1}\hat{\mathbb{E}}\left[\int_0^T |X_t|^p|\eta_t - \eta_t^n|^p \mathrm{d}t\right] + 2^{p-1}\hat{\mathbb{E}}\left[\int_0^T |X_t - X_t^n|^p|\eta_t^n|^p \mathrm{d}t\right].$$

By Proposition 3.10, the first term on the right-hand side tends to 0. Since η^n is uniformly bounded, the second term also tends to 0. □

Remark 3.12. It is easy to prove that if $\eta \in M_*^2(0, T)$, then $\int_0^\cdot \eta_s \mathrm{d}B_s \in M_*^2(0, T)$.

Let $\mathbf{a} = (a_1, \ldots, a_d)^T$ and $\bar{\mathbf{a}} = (\bar{a}_1, \ldots, \bar{a}_d)^T$ be two given vectors in $\mathbb{R}^d$. We can define

$$\langle B^{\mathbf{a}}\rangle_t := (B_t^{\mathbf{a}})^2 - 2\int_0^t B_s^{\mathbf{a}} \mathrm{d}B_s^{\mathbf{a}},$$

where $\langle B^{\mathbf{a}}\rangle$ is called the quadratic variation process of $B^{\mathbf{a}}$. We can also define mutual variation process by

$$\begin{aligned}\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_t &:= \frac{1}{4}[\langle B^{\mathbf{a}} + B^{\bar{\mathbf{a}}}\rangle_t - \langle B^{\mathbf{a}} - B^{\bar{\mathbf{a}}}\rangle_t] \\ &= \frac{1}{4}[\langle B^{\mathbf{a}+\bar{\mathbf{a}}}\rangle_t - \langle B^{\mathbf{a}-\bar{\mathbf{a}}}\rangle_t].\end{aligned}$$

Itô's integral with respect to $\langle B^{\mathbf{a}}\rangle$ or $\langle B^i, B^j\rangle$ can be similarly defined (see the Appendix of [15]). For the properties of above processes and integrals, we refer the reader to Peng [21–25].

4. Itô's integral with stopping times

In this section, we study Itô's integral on an interval $[0, \tau]$, where τ is a stopping time. Readers can see that, thanks to Propositions 3.8, 3.10 and 3.11, the techniques used in this section are very similar to the classical situation.

Definition 4.1. A stopping time τ relative to the filtration $(\mathcal{F}_t)$ (see Section 2) is a map on Ω with values in $[0, T]$, such that for every $t \leq T$,

$$\{\tau \leq t\} \in \mathcal{F}_t.$$

Lemma 4.2. *For each stopping time τ, we have $\mathbf{I}_{[0,\tau]}(\cdot)\eta \in M_*^p(0, T)$, for each $\eta \in M_*^p(0, T)$.*

Proof. For the given stopping time τ, let

$$\tau_n = \sum_{k=0}^{2^n-1} \frac{(k+1)T}{2^n}\mathbf{I}_{\left[\frac{kT}{2^n} < \tau \leq \frac{(k+1)T}{2^n}\right]}.$$

Then we have $2^{-n}T \geq \tau_n - \tau \geq 0$. It is clear that, for $m \geq n$,

$$\hat{\mathbb{E}}\left[\int_0^T |\mathbf{I}_{[0,\tau_n]}(t) - \mathbf{I}_{[0,\tau_m]}(t)|\mathrm{d}t\right] \leq \hat{\mathbb{E}}\left[\int_0^T |\mathbf{I}_{[0,\tau_n]}(t) - \mathbf{I}_{[0,\tau]}(t)|\mathrm{d}t\right]$$
$$= \hat{\mathbb{E}}[\tau_n - \tau] \leq 2^{-n}T.$$

By Proposition 3.11, $\{\mathbf{I}_{[0,\tau_n]}\eta\}_{n=1}^{\infty}$ is in $M_*^p(0,T)$. From Proposition 3.10 it follows that this sequence is a Cauchy one in $M_*^p(0,T)$. Thus $\mathbf{I}_{[0,\tau]}\eta \in M_*^p(0,T)$. □

Lemma 4.3. *For each stopping time τ and $\eta \in M_*^p(0,T)$, we have*

$$\int_0^{t\wedge\tau} \eta_s \mathrm{d}B_s^{\mathbf{a}} = \int_0^t \mathbf{I}_{[0,\tau]}(s)\eta_s \mathrm{d}B_s^{\mathbf{a}}, \quad \text{q.s.} \tag{9}$$

Proof. For each $n \in \mathbb{N}$, let

$$\tau_n := \sum_{k=1}^{2^n} \mathbf{I}_{A_n^k} t_n^k$$

where $t_n^k = k2^{-n}t$, $A_n^k = [t_n^{k-1} < t \wedge \tau \leq t_n^k]$ for $k < 2^n$, and $A_n^{2^n} = [\tau \geq t]$. $\{\tau_n\}_{n=1}^{\infty}$ is a decreasing sequence of stopping times which converges to $t \wedge \tau$ q.s. We first prove that

$$\int_{\tau_n}^t \eta_s \mathrm{d}B_s^{\mathbf{a}} = \int_0^t \mathbf{I}_{[\tau_n,t]}(s)\eta_s \mathrm{d}B_s^{\mathbf{a}}, \quad \text{q.s.} \tag{10}$$

By Proposition 3.5, we have

$$\int_{\tau_n}^t \eta_s \mathrm{d}B_s^{\mathbf{a}} = \int_{\sum_{k=1}^{2^n} \mathbf{I}_{A_n^k} t_n^k}^t \eta_s \mathrm{d}B_s^{\mathbf{a}} = \sum_{k=1}^{2^n} \mathbf{I}_{A_n^k} \int_{t_n^k}^t \eta_s \mathrm{d}B_s^{\mathbf{a}}$$
$$= \sum_{k=1}^{2^n} \int_{t_n^k}^t \mathbf{I}_{A_n^k} \eta_s \mathrm{d}B_s^{\mathbf{a}}$$
$$= \int_0^t \sum_{k=1}^{2^n} \mathbf{I}_{[t_n^k,t]}(s)\mathbf{I}_{A_n^k}\eta_s \mathrm{d}B_s^{\mathbf{a}}, \quad \text{q.s.,}$$

from which (10) follows. We thus have

$$\int_0^{\tau_n} \eta_s \mathrm{d}B_s^{\mathbf{a}} = \int_0^t \mathbf{I}_{[0,\tau_n]}(s)\eta_s \mathrm{d}B_s^{\mathbf{a}}, \quad \text{q.s.}$$

Observe that $0 \leq \tau_n - \tau_m \leq \tau_n - t \wedge \tau \leq 2^{-n}t$ for $n \leq m$. From this with Proposition 3.10 it follows that $\mathbf{I}_{[0,\tau_n]}\eta$ converges in $M_*^2(0,T)$ to $\mathbf{I}_{[0,\tau\wedge t]}\eta$ and thus $\mathbf{I}_{[0,\tau\wedge t]}\eta \in M_*^2(0,T)$. Consequently,

$$\lim_{n\to\infty} \int_0^{\tau_n} \eta_s \mathrm{d}B_s^{\mathbf{a}} = \int_0^{t\wedge\tau} \eta_s \mathrm{d}B_s^{\mathbf{a}}, \quad \text{q.s.}$$

and (9) holds as well. □

Definition 4.4. Let $p > 0$ be fixed. A stochastic process $(\eta_t)_{t\geq 0}$ is said to be in $M_\omega^p(0,T)$ if there exists a sequence of increasing stopping times $\{\sigma_m\}_{m=1}^\infty$, with $\sigma_m \uparrow T$, q.s., such that $\eta \mathbf{I}_{[0,\sigma_m]} \in M_*^p(0,T)$ and

$$\int_0^T |\eta_s|^p \mathrm{d}s < \infty, \quad \text{q.s.}$$

Remark 4.5. In the rest of this paper, the notation $\{\sigma_m\}_{m=1}^\infty$ is used to denote the sequence of the corresponding process $\eta \in M_\omega^p(0,T)$. Let $\{\tau_m\}_{m=1}^\infty$ be another sequence of increasing stopping times with $\tau_m \uparrow T$, q.s. Then it is easy to check that $\eta \mathbf{I}_{[0,\sigma_m\wedge\tau_m]} \in M_*^p(0,T)$ and $\sigma_m \wedge \tau_m \uparrow T$, q.s. Thus we can as well use $\sigma_m \wedge \tau_m$ in place of σ_m. For example, when we consider two processes $\eta, \bar{\eta} \in M_\omega^p(0,T)$ with two sequences of stopping times $\{\sigma_m\}_{m=1}^\infty$ and $\{\bar{\sigma}_m\}_{m=1}^\infty$, we may only use one sequence $\{\sigma_m \wedge \bar{\sigma}_m\}_{m=1}^\infty$ for both η and $\bar{\eta}$.

Proposition 4.6. *Let $\eta \in M_\omega^1(0,T)$ be given and let*

$$\tau_n = \inf\left\{t \geq 0, \int_0^t |\eta_s| \mathrm{d}s \geq n\right\} \wedge \sigma_n.$$

Then $\eta \mathbf{I}_{[0,\tau_n]} \in M_*^1(0,T)$, $\int_0^t \mathbf{I}_{[0,\tau_n]}(s)\eta_s \mathrm{d}s$ and $\int_0^t \mathbf{I}_{[0,\tau_n]}(s)\eta_s \mathrm{d}\langle B^{\mathbf{a}}\rangle_s$ are well-defined processes which are continuous on $[0,T]$ quasi-surely.

The proof is similar to that of the following proposition.

Proposition 4.7. *Let $\eta \in M_\omega^2(0,T)$ and $\tau_n = \inf\{t \geq 0, \int_0^t |\eta_s|^2 \mathrm{d}s \geq n\} \wedge \sigma_n$. Then $\eta \mathbf{I}_{[0,\tau_n]} \in M_*^2(0,T)$ and the stochastic process $(\int_0^t \eta_s \mathrm{d}B_s^{\mathbf{a}})_{t\in[0,T]}$ is a well-defined quasi-surely continuous process on $[0,T]$.*

Proof. Since, for each $n = 1, 2, \ldots$, $\eta \mathbf{I}_{[0,\tau_n]} \in M_*^2(0,T)$, Itô's integral $\int_0^t \mathbf{I}_{[0,\tau_n]}(s)\eta_s \mathrm{d}B_s^{\mathbf{a}}$ is well defined. On the other hand, on the subset $\Omega_n = \{\tau_n = T\}$ and for each $m > n$, we have $\tau_m = \tau_n = T$. Then

$$\lim_{m\to\infty} \mathbf{I}_{\Omega_n} \int_0^{\tau_m\wedge t} \eta_s \mathrm{d}B_s^{\mathbf{a}} = \mathbf{I}_{\Omega_n} \int_0^{\tau_n\wedge t} \eta_s \mathrm{d}B_s^{\mathbf{a}} = \mathbf{I}_{\Omega_n} \int_0^t \eta_s \mathrm{d}B_s^{\mathbf{a}}, \quad t \in [0,T].$$

Thus on Ω_n the process $(\int_0^t \eta_s \mathrm{d}B_s^{\mathbf{a}})_{t\in[0,T]}$ is a well-defined process which is continuous in t quasi-surely. Since $\Omega_n \uparrow \bar{\Omega} \subset \Omega$, with $\hat{c}(\bar{\Omega}^c) = 0$. It follows $(\int_0^t \eta_s \mathrm{d}B_s^{\mathbf{a}})_{t\in[0,T]}$ can be a well-defined process which is continuous in t quasi-surely. $\square$

5. Itô's formula

Lemma 5.1. *We assume that $\Phi \in C^2(\mathbb{R}^n)$ and all first and second order derivatives of Φ are in $C_{b,\mathrm{Lip}}(\mathbb{R}^n)$. Let $X = (X^1, \ldots, X^n)$ and*

$$X_t^\nu = X_0^\nu + \int_0^t \alpha_s^\nu \mathrm{d}s + \int_0^t \eta_s^{\nu ij} \mathrm{d}\langle B^i, B^j\rangle_s + \int_0^t \beta_s^{\nu j} \mathrm{d}B_s^j$$

where, for $\nu = 1, \ldots, n$, $i, j = 1, \ldots, d$, α^ν, $\eta^{\nu ij}$ and $\beta^{\nu j}$ are bounded elements in $M_*^2(0,T)$. Then for each $t \in [0,T]$, we have, quasi-surely,

$$\Phi(X_t) - \Phi(X_0) = \int_0^t \partial_{x^\nu}\Phi(X_u)\beta_u^{\nu j} \mathrm{d}B_u^j + \int_0^t \partial_{x^\nu}\Phi(X_u)\alpha_u^\nu \mathrm{d}u$$

$$+\int_0^t \left[\partial_{x^\nu}\Phi(X_u)\eta_u^{\nu ij}+\frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_u)\beta_u^{\mu i}\beta_u^{\nu j}\right]\mathrm{d}\langle B^i,B^j\rangle_u. \tag{11}$$

Here and in the rest of this paper, we use the Einstein notation of summing over repeated indices μ,ν,i and j.

The proof is very similar to those of Lemma 46 and Proposition 48 in [22] (see Appendix of [15] for the complete proof).

Lemma 5.2. *Let $\Phi \in C^2(\mathbb{R}^n)$ and its first and second derivatives are in $C_{b,\,\mathrm{Lip}}(\mathbb{R}^n)$. Let $X=(X^1,\dots,X^n)$ be an n-dimensional process on $[0,T]$ with the form $X_t^\nu = X_0^\nu+\int_0^t\alpha_s^\nu \mathrm{d}s+\int_0^t\eta_s^{\nu ij}\mathrm{d}\langle B^i,B^j\rangle_s+\int_0^t\beta_s^{\nu j}\mathrm{d}B_s^j$, where $\alpha^\nu,\eta^{\nu ij}\in M_*^1(0,T)$, $\beta^{\nu j}\in M_*^2(0,T)$. Then for each $t\in[0,T]$, we have, quasi-surely,*

$$\Phi(X_t)-\Phi(X_0)=\int_0^t\partial_{x^\nu}\Phi(X_u)\beta_u^{\nu j}\mathrm{d}B_u^j+\int_0^t\partial_{x^\nu}\Phi(X_u)\alpha_u^\nu\mathrm{d}u$$
$$+\int_0^t \left[\partial_{x^\nu}\Phi(X_u)\eta_u^{\nu ij}+\frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_u)\beta_u^{\mu i}\beta_u^{\nu j}\right]\mathrm{d}\langle B^i,B^j\rangle_u. \tag{12}$$

Proof. For simplicity, we only state for the case of $n=1$ and $d=1$. Let $\alpha^{(k)}$, $\beta^{(k)}$ and $\eta^{(k)}$ be bounded processes such that, as $k\to\infty$,

$$\alpha^{(k)}\to\alpha,\qquad \eta^{(k)}\to\eta \quad \text{in } M_*^1(0,T) \quad \text{and} \quad \beta^{(k)}\to\beta \quad \text{in } M_*^2(0,T)$$

and let

$$X_t^{(k)}=X_0+\int_0^t\alpha_s^{(k)}\mathrm{d}s+\int_0^t\eta_s^{(k)}\mathrm{d}\langle B\rangle_s+\int_0^t\beta_s^{(k)}\mathrm{d}B_s.$$

Then we have

$$\lim_{k\to\infty}\hat{\mathbb{E}}[\sup_{0\le t\le T}|X_t-X_t^{(k)}|]=0 \quad \text{and} \quad \lim_{k\to\infty}\hat{\mathbb{E}}[\sup_{0\le t\le T}|\Phi(X_t)-\Phi(X_t^{(k)})|]=0.$$

We see that

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\int_0^T|\partial_x\Phi(X_t^{(k)})\beta_t^{(k)}-\partial_x\Phi(X_t)\beta_t|^2\mathrm{d}t\right]\\
&\le 2\hat{\mathbb{E}}\left[\int_0^T|\partial_x\Phi(X_t^{(k)})\beta_t^{(k)}-\partial_x\Phi(X_t^{(k)})\beta_t|^2\mathrm{d}t\right]\\
&\quad+2\hat{\mathbb{E}}\left[\int_0^T|\partial_x\Phi(X_t^{(k)})\beta_t-\partial_x\Phi(X_t)\beta_t|^2\mathrm{d}t\right]\\
&\le C\hat{\mathbb{E}}\left[\int_0^T|\beta_t^{(k)}-\beta_t|^2\mathrm{d}t\right]+2\hat{\mathbb{E}}\left[\int_0^T|\beta_t|^2|\partial_x\Phi(X_t^{(k)})-\partial_x\Phi(X_t)|^2\mathrm{d}t\right].
\end{aligned}$$

But we have $\sup_{0\le t\le T}|\partial_x\Phi(X_t^{(k)})-\partial_x\Phi(X_t)|\le C$ and

$$\hat{\mathbb{E}}\left[\int_0^T|\partial_x\Phi(X_t^{(k)})-\partial_x\Phi(X_t)|^2\mathrm{d}t\right]\to 0, \quad \text{as } k\to\infty.$$

Thus we can apply Proposition 3.10 to prove that $\partial_x\Phi(X^{(k)})\beta^{(k)}\to\partial_x\Phi(X)\beta$ in $M_*^2(0,T)$. Similarly, $\partial_x\Phi(X^{(k)})\alpha^{(k)}\to\partial_x\Phi(X)\alpha$, $\partial_x\Phi(X^{(k)})\eta^{(k)}\to\partial_x\Phi(X)\eta$ and $\partial^2_{xx}\Phi(X^{(k)})(\beta^{(k)})^2\to$

$\partial^2_{xx}\Phi(X)\beta^2$ in $M^1_*(0,T)$. But from the above lemma we have

$$\Phi(X_t^{(k)}) - \Phi(X_0^{(k)}) = \int_0^t \partial_x \Phi(X_u^{(k)})\beta_u^{(k)}\mathrm{d}B_u + \int_0^t \partial_x \Phi(X_u^{(k)})\alpha_u^{(k)}\mathrm{d}u + \int_0^t \left[\partial_x \Phi(X_u^{(k)})\eta_u^{(k)} + \frac{1}{2}\partial^2_{xx}\Phi(X_u^{(k)})(\beta_u^{(k)})^2\right]\mathrm{d}\langle B\rangle_u.$$

We then can pass to the limit on both sides of the above equality, as $k \to \infty$, to obtain (12). □

Lemma 5.3. *Let X be given as in the above lemma and let $\Phi \in C^{1,2}([0,T]\times\mathbb{R}^n)$ such that Φ, $\partial_t\Phi$, $\partial_x\Phi$ and $\partial^2_{xx}\Phi$ are bounded and uniformly continuous on $[0,T]\times\mathbb{R}^n$. Then we have*

$$\Phi(t,X_t) - \Phi(0,X_0) = \int_0^t \partial_{x^\nu}\Phi(u,X_u)\beta_u^{\nu j}\mathrm{d}B_u^j + \int_0^t [\partial_t\Phi(u,X_u) + \partial_{x^\nu}\Phi(u,X_u)\alpha_u^\nu]\mathrm{d}u + \int_0^t \left[\partial_{x^\nu}\Phi(u,X_u)\eta_u^{\nu ij} + \frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(u,X_u)\beta_u^{\mu i}\beta_u^{\nu j}\right]\mathrm{d}\langle B^i,B^j\rangle_u.$$

Proof. We can take $\{\Phi_k\}_{k=1}^\infty$ such that, for each k, Φ_k and all its first order and second order derivatives are in $C_{b,\,\mathrm{Lip}}([0,T]\times\mathbb{R}^n)$ and such that, as $k\to\infty$, Φ_k, $\partial_t\Phi_k$, $\partial_x\Phi_k$ and $\partial^2_{xx}\Phi_k$ converge respectively to Φ, $\partial_t\Phi$, $\partial_x\Phi$ and $\partial^2_{xx}\Phi$ uniformly on $[0,T]\times\mathbb{R}$. We then use the above Itô's formula to $\Phi_k(X_t^0, X_t)$, with $Y_t = (X_t^0, X_t)$, where $X_t^0 \equiv t$:

$$\Phi_k(t,X_t) - \Phi_k(0,X_0) = \int_0^t \partial_{x^\nu}\Phi_k(u,X_u)\beta_u^{\nu j}\mathrm{d}B_u^j + \int_0^t [\partial_t\Phi_k(u,X_u) + \partial_{x^\nu}\Phi_k(u,X_u)\alpha_u^\nu]\mathrm{d}u + \int_0^t \left[\partial_{x^\nu}\Phi_k(u,X_u)\eta_u^{\nu ij} + \frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi_k(u,X_u)\beta_u^{\mu i}\beta_u^{\nu j}\right]\mathrm{d}\langle B^i,B^j\rangle_u.$$

It follows that, as $k\to\infty$, we have uniformly

$$|\partial_{x^\nu}\Phi_k(u,X_u) - \partial_{x^\nu}\Phi(u,X_u)| \to 0, \qquad |\partial^2_{x^\mu x^\nu}\Phi_k(u,X_u) - \partial^2_{x^\mu x^\nu}\Phi(u,X_u)| \to 0,$$
$$|\partial_t\Phi_k(u,X_u) - \partial_t\Phi(u,X_u)| \to 0.$$

We then can apply the above lemma to $\Phi_k(t,X_t) - \Phi_k(0,X_0)$ and pass to the limit as $k\to\infty$ to obtain the desired result. □

Theorem 5.4. *Let $\Phi \in C^{1,2}([0,T]\times\mathbb{R})$ and*

$$X_t^\nu = X_0^\nu + \int_0^t \alpha_s^\nu \mathrm{d}s + \int_0^t \eta_s^{\nu ij}\mathrm{d}\langle B^i,B^j\rangle_s + \int_0^t \beta_s^{\nu j}\mathrm{d}B_s^j,$$

where $\alpha^\nu, \eta^{\nu ij} \in M^1_\omega(0,T)$, $\beta^{\nu j} \in M^2_\omega(0,T)$. Then for each $t\in[0,T]$, we have, quasi-surely,

$$\Phi(t,X_t) - \Phi(0,X_0) = \int_0^t \partial_{x^\nu}\Phi(u,X_u)\beta_u^{\nu j}\mathrm{d}B_u^j + \int_0^t [\partial_t\Phi(u,X_u) + \partial_{x^\nu}\Phi(u,X_u)\alpha_u^\nu]\mathrm{d}u + \int_0^t \left[\partial_{x^\nu}\Phi(u,X_u)\eta_u^{\nu ij} + \frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(u,X_u)\beta_u^{\mu i}\beta_u^{\nu j}\right]\mathrm{d}\langle B^i,B^j\rangle_u.$$

Proof. For simplicity, we only prove the case of $n = d = 1$. We set, for $k = 1, 2, \ldots,$

$$\gamma_t = |X_t| + \int_0^t (|\beta_u|^2 + |\alpha_u| + |\eta_u|)\mathrm{d}u$$

and $\tau_k := \inf\{t \geq 0, \gamma_t \geq k\} \wedge \sigma_k$. Let Φ_k be a $C^{1,2}$-function on $[0, T] \times \mathbb{R}^n$ such that $\Phi_k, \partial_t \Phi_k, \partial_x \Phi_k$ and $\partial^2_{xx} \Phi_k$ are bounded uniformly continuous and such that $\Phi_k = \Phi$, for $|x| \leq 2k,\ t \in [0, T]$. It is clear that

$$\mathbf{I}_{[0,\tau_k]}\beta \in M^2_*(0, T), \qquad \mathbf{I}_{[0,\tau_k]}\alpha, \qquad \mathbf{I}_{[0,\tau_k]}\eta \in M^1_*(0, T)$$

and we have quasi-surely

$$X_{t\wedge\tau_k} = X_0 + \int_0^t \alpha_s \mathbf{I}_{[0,\tau_k]}\mathrm{d}s + \int_0^t \eta_s \mathbf{I}_{[0,\tau_k]}\mathrm{d}\langle B\rangle_s + \int_0^t \beta_s \mathbf{I}_{[0,\tau_k]}\mathrm{d}B_s.$$

We then can apply the above lemma to $\Phi_k(t, X_{t\wedge\tau_k})$ to obtain

$$\begin{aligned}\Phi(t, X_{t\wedge\tau_k}) - \Phi(0, X_0) &= \int_0^t \partial_x \Phi(u, X_u)\beta_u \mathbf{I}_{[0,\tau_k]}\mathrm{d}B_u \\ &\quad + \int_0^t [\partial_t \Phi(u, X_u) + \partial_x \Phi(u, X_u)\alpha_u]\mathbf{I}_{[0,\tau_k]}\mathrm{d}u \\ &\quad + \int_0^t \left[\partial_x \Phi(u, X_u)\eta_u \mathbf{I}_{[0,\tau_k]} + \frac{1}{2}\partial^2_{xx}\Phi(u, X_u)\beta^2_u \mathbf{I}_{[0,\tau_k]}\right]\mathrm{d}\langle B\rangle_u, \quad \text{q.s.}\end{aligned}$$

Passing to the limit as $k \to \infty$ and noticing that X_t is quasi-surely continuous in t, we then obtain the desired result. $\square$

The following example solves the problem we raised at the beginning of the introduction (see (1)) for a 1-dimensional G-Brownian motion B.

Example 5.5. For a given $\Phi \in C^2(\mathbb{R})$, we have

$$\Phi(B_t) - \Phi(B_0) = \int_0^t \Phi_x(B_s)\mathrm{d}B_s + \frac{1}{2}\int_0^t \Phi_{xx}(B_s)\mathrm{d}\langle B\rangle_s.$$

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References

[1] M. Avellaneda, A. Levy, A. Paras, Pricing and hedging derivative securities in markets with uncertain volatilities, Applied Mathematical Finance 2 (1995) 73–88.
[2] Z. Chen, L. Epstein, Ambiguity, risk and asset returns in continuous time, Econometrica 70 (4) (2002) 1403–1443.
[3] P. Cheridito, H.M. Soner, N. Touzi, N. Victoir, Second order backward stochastic differential equations and fully non-linear parabolic PDEs, Communications on Pure & Applied Mathematics 60 (7) (2007) 1081–1110.
[4] G. Choquet, Theory of capacities, Annales de l'Institut Fourier 5 (1953) 131–295.
[5] L. Denis, M. Hu, S. Peng, Function spaces and capacity related to a sublinear expectation: application to G-Brownian motion paths, Potential Analysis (2010) 1–23.

[6] L. Denis, C. Martini, A theoretical framework for the pricing of contingent claims in the presence of model uncertainty, The Annals of Applied Probability 16 (2) (2006) 827–852.

[7] M. Frittelli, E. Rossaza Gianin, Dynamic Convex Risk Measures, Risk Measures for the 21st Century, J. Wiley, 2004, pp. 227–247.

[8] F.Q. Gao, Pathwise properties and homeomorphic flows for stochastic differential equations driven by G-Brownian motion, Stochastic Processes and their Applications 119 (10) (2009) 3356–3382.

[9] F.Q. Gao, H. Jiang, Large deviations for stochastic differential equations driven by G-Brownian motion, Stochastic Processes and their Applications 120 (11) (2010) 2212–2240.

[10] M. Hu, Explicit solutions of G-heat equation with a class of initial conditions by G-Brownian motion, 2008. arXiv:0802.1240v1 [math.PR].

[11] M. Hu, S. Peng, G-Levy processes under sublinear expectations, 2009. Preprint in: arXiv:0911.3533v1.

[12] M. Hu, S. Peng, On representation theorem of G-expectations and paths of G-Brownian motion, Acta Mathematica Sinica, English Series 25 (3) (2009) 1–8.

[13] M. Liao, G-Browian motion in Lie groups, 2009. Preprint.

[14] Q. Lin, Some properties of G-expectation, 2008. Preprint.

[15] X. Li, S. Peng, Stopping times and related Itô's calculus with G-Brownian motion, 2009. arXiv:0910.3871v2 [math.PR].

[16] T. Lyons, Uncertain volatility and the risk free synthesis of derivatives, Applied Mathematical Finance 2 (11) (1995) 7–133.

[17] K. Magali, Etude des Modèles non dominé en Mathématiques Financières, Thèse, Université d'Evry Val d'Esonne, 2008.

[18] S. Peng, Backward SDE and related g-expectations, in: Backward Stochastic Differential Equations, in: Pitman Research Notes in Math. Series, vol. 364, El Karoui Mazliak edit, 1997, pp. 141–159.

[19] S. Peng, Filtration consistent nonlinear expectations and evaluations of contingent claims, Acta Mathematicae Applicatae Sinica (English Series) 20 (2) (2004) 1–24.

[20] S. Peng, Nonlinear expectations and nonlinear Markov chains, Chinese Annals of Mathematics 26B (2) (2005) 159–184.

[21] S. Peng, G-Expectation, G-Brownian motion and related stochastic calculus of Itô's type, in: Proceedings of the 2005 Abel Symposium, Benth et. al., Springer-Verlag, January 3, 2006, pp. 541–567, Preprint version in: arXiv:math.PR/0601035v1.

[22] S. Peng, Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation, Stochastic Processes and their Applications 118 (12) (2008) 2223–2253.

[23] S. Peng, Survey on normal distributions, central limit theorem, Brownian motion and the related stochastic calculus under sublinear expectations, Science in China Series A: Mathematics 52 (7) (2009) 1391–1411.

[24] S. Peng, Nonlinear expectations and stochastic calculus under uncertainty—with robust central limit theorem and G-Brownian motion, 2010. arXiv:1002.4546v1.

[25] S. Peng, Backward stochastic differential equation, nonlinear expectation and their applications, in: Proceedings of the International Congress of Mathematicians, Hyderabad, India, 2010.

[26] J. Xu, Z. Zhang, Martingale characterization of G-Brownian motion, Stochastic Processes and their Applications 119 (1) (2009) 232–248.

[27] B. Zhang, J. Xu, D. Kannan, Extension and application of Itô's formula under G-framework, Stochastic Analysis and Applications 28 (2010) 322–349.